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POOLING: ASSIGNMENT WITH PROPERTY RIGHTS

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ABSTRACT: Pooling situations and two cooperative games related to them are considered.
Pooling situations are assignment problems under the presence of property or option rights. The Owen vectors of the two cooperative games related to a pooling situation exhibit a kind of consistency which makes both methods of evaluating the pooling situation equivalent.
An example shows that this does not hold for the core, the Shapley value, the nucleolus and the τ-value.

1. INTRODUCTION

The subject of this paper is the question what can be gained by pooling option or property rights and how the joint profit, earned by this way of cooperation, should be divided among the pooling partners. A pooling situation occurs, when owners of mutually replaceable commodities are willing to give up, for the time being, their property rights in order to reach a more profitable reallocation of the commodities. In this context replaceability includes that it is possible to compensate a person by means of money for the replacement of a quantity of one commodity by the same quantity of another commodity. This places the paper into the domain of transferable utility. Once, when the optimal allocation has been found, the joint profit is divided in a way which takes the original property rights into account. In this paper it does not matter whether the commodities considered are infinitely divisible or are only available in indivisible quantities.

Permutation games as studied in [3] and [18] provide good examples of pooling situations. They describe a situation in which n persons all have one job to be processed and possess one machine on which each job can be processed. No machine is allowed to process more than one job. If player i processes his job on the machine of player j, the earnings will be $E_{ij} \in \mathbb{R}_+$. The players look for a permutation of the machines

which maximizes the total earnings and so, in fact, they are pooling their machines.
Another example is provided by a firm with several divisions, each of which has its own secretarial staff. By pooling the secretaries, there can be made a more efficient use of the services of the secretaries.

Pooling situations give rise to two rather independent problems. The first problem asks for an optimal allocation of the commodities involved. This turns out to be a real- or integer-valued linear programming problem (L.P. problem). Since real-valued linear programs attain their optimal value also in extreme points of the set of feasible points, it is interesting to know these extreme points exactly. In section 2 we shall give a procedure to find, in the case of pooling situations, all the extreme points of the feasible set. The second problem, how to divide the joints profit, we attack with tools provided by cooperative game theory. To each pooling situation we define a cooperative game (with side payments) in such a way that the worth of a coalition is the maximal earnings which the coalition can gain by pooling only the commodities of its members. The cooperative game, obtained in this way, turns out to have a non-empty core. Since core elements are defined by the property that no subcoalition can strictly gain by dividing the maximal earnings of the coalition over the members, they provide an answer to the second problem. But we can do better; after minor changes the cooperative games related to pooling situations fit perfectly well into the framework of linear production games as initiated by Owen in [11] (for more general results see Granot [6] and Dubey/Shapley [4]). Owen proves the non-emptiness of the core of production games by showing that divisions of the joint profit according to the value of the resources of the players under so-called shadow prices are core elements. We shall call the core elements obtained by the use of shadow prices, the Owen vectors of the pooling situation and these will have our special attention. There can also be defined another cooperative game related to pooling situations. In fact this game is a generalization of the assignment game of Shapley/ Shubik [16]. The player set of this game consists of the pooling partners as well as the pooled commodities.[1] The worth of a coalition $S \subset N \cup M$ is the maximal earnings which S can obtain by cooperation under complete neglection of the ownership relations. The second cooperative game related to a pooling situation can again be seen as a production game but now core elements (or Owen vectors) attributes the joint profit to the pooling partners as well as to the pooled commodities. A main result of this paper states a kind of consistency between the Owen vectors of the both cooperative games related to pooling situations. An example in section 3 will show that more familiar solution concepts as core, Shapley value, nucleolus and τ-value do, in general, not have this property.

The paper is organized as follows. After two examples of pooling situations, we give in section 2 a formal definition of a pooling situation and introduce the two cooperative games, related to it. Further we shall discuss briefly the extreme points of the set of feasible reallocations. A forthcoming paper will give a more thorough discussion of this subject. A direct proof of the non-emptiness of both the cores is also provided.

In section 3 we recall the results of Owen on production games, prove the consistency of the Owen vectors of the two cooperative games related to pooling situations and provide the counter-example. In section 4 we discuss some topics which are related to our problem.

2. POOLING SITUATIONS AND RELATED COOPERATIVE GAMES

Before giving a formal definition of a pooling situation, we start with two examples.

Example 2.1. (Hotel Reservation) Before the season travel

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[1] This approach is less peculiar than it may look at first glance if we remember that in the second example above the commodities were (the services of) secretaries.

agencies $T_1,\dots,T_n$ make reservations for rooms in the hotels $H_1,\dots,H_m$ of a holidays resort. Let O_{ij} denote the number of rooms in hotel H_j for which travel agency T_i made reservations. This means: during whole the season travel agency T_i has the right to lodge up to O_{ij} of his clients in hotel H_j and hotel H_j has the obligation to take in up to that number of guests from T_i. Let $s_j = \sum_{i\in N} O_{ij}$ be the total number of reserved rooms in hotel H_j. Further, let $E_{ij} \in \mathbb{R}_+$ be the net earnings which T_i can achieve by lodging one of his clients in hotel H_j (for a week e.g.). In order to absorb fluctuations in the demand for rooms during the season, the travel agencies decide to pool their reservations i.e. in lodging their clients the travel agencies need not consider which agency made reservations in which hotel. They simply lodge their clients such that the joint earnings are maximal. Suppose that in some week travel agency T_i has d_i guests to be lodged. If for $i \in N = \{1,\dots,n\}$ and $j \in M = \{1,\dots,m\}$ the number of guests, which hotel H_j takes in from travel agency T_i, is denoted by X_{ij}, then the $N\times M$-matrix X has to satisfy the following constraints to be feasible:

$$X_{ij} \geq 0; \quad \sum_{j\in M} X_{ij} \leq d_i \text{ and } \sum_{i\in N} X_{ij} \leq s_j, \text{ for all } i \in N \text{ and all } j \in M.$$

In this case, moreover $X_{ij} \in Z_+$ where Z_+ is the set of non-negative integers. So,we have to find an $N\times M$-matrix $X \in Z_+^{N\times M}$ such that

$$1_N X \leq s \quad \text{and} \quad X\, 1_M \leq d \tag{2.1}$$

which maximizes $X*E = \sum_{i\in N, j\in M} X_{ij}E_{ij}$ where 1_N is the vector $(1,\dots,1) \in \mathbb{R}^N$, $s = (s_j)_{j\in M}$ and $d = (d_i)_{i\in N}$.

Example 2.2. (Tea Market) Let us consider the situation in which traders in tea $T_1,\dots,T_n$ have taken options from tea producing companies $C_1,\dots,C_m$ for the quantities of tea $O_{ij} \in \mathbb{R}_+$ they want to purchase. Let $E_{ij} \in \mathbb{R}_+$ be the net profit which trader T_i can gain from one unit of tea,

purchased from company C_j.
At the end of the option period each trader knows how much tea he wants actually to purchase. Let d_i be this quantity for each $i \in N$. By pooling their option rights the traders have the opportunity to raise their joint profit. So, they look for a real-valued matrix $X \in \mathbb{R}_+^{N\times M}$ satisfying the constraints (2.1) which maximizes the joint profit $X*E$. The number X_{ij} denotes, for each $i \in N$ and $j \in M$, the quantity of tea purchased by trader T_i from company C_j.

Definition 2.3. A pooling situation is a six-tuple $\langle N,M,s,d,O,E\rangle$ where N and M are finite sets with n and m elements respectively, $s \in \mathbb{R}_+^M$, $d \in \mathbb{R}_+^N$, $E \in \mathbb{R}_+^{N\times M}$ and $O \in \mathbb{R}_+^{N\times M}$ with $1_N O = s$.
The set of feasible distributions of a pooling situation $\langle N,M,s,d,O,E\rangle$ consists of the $N\times M$-matrices X with the properties $X \geq 0$, $1_N X \leq s$, $X 1_M \leq d$.
We denote this set by Θ. If we, moreover, ask for integer-valued matrices X, this set is denoted Θ^*.
The first problem, we mentioned in the introduction, is to find a feasible distribution $X \in \Theta$ (or Θ^*) which maximizes $X*E$.
To find an optimal solution of this problem we can use what Kuhn [10] baptized the Hungarian method after rediscovery of the results of König [9] and Egervàry [5]. We can also describe a procedure which generates all the extreme points of the set Θ of feasible distributions. This proceeds as follows: Take a subset $K \subset N$ and a subset $L \subset M$ and arrange the entries of $K\times L$ in a linear order. Put on the entries of $K\times L$ successively, according to the given order, the maximal number which satisfies the constraints up to that moment. Put zeroes outside of $K\times L$. In this way we get all extreme points of Θ by varying the set $K\times L$ and the order in which the matrix is filled up. More details and the proof of this statement will be the subject of a forthcoming paper. An example may elucidate the procedure.

Example 2.4. Let $N = \{1,2,3\}$, $M = \{1,2,3,4\}$. Further $d = (3,5,6)$ and $s = (2,2,8,1)$. Suppose $K = \{1,2\}$, $L = \{1,2,3\}$ and the linear order of $K \times L$ is

$$(1,1) < (2,2) < (2,1) < (1,3) < (1,2) < (2,3).$$

This generates the matrix

$$\begin{array}{cc} & d \\ X = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} 3 \\ 5 \\ 6 \end{array} \\ s \quad \begin{array}{cccc} 2 & 2 & 8 & 1 \end{array} & \end{array}$$

as can easily be verified.

Corollary 2.5.

(i) If $d \in Z_+^N$ and $s \in Z_+^M$ (the constraints are integer-valued) then all extreme points of Θ are integer-valued and Θ is the convex hull of Θ^*.

(ii) Since every step in the filling-up process which generates value $x_{ij} > 0$ increases the number of completed rows or columns, there can be at most $n+m-1$ entries with a positive value in X.

To attack the second problem, to find a reasonable division of the joint profit we introduce a cooperative game (with side payments). A cooperative game (with side payments) consists of a player set P and a (characteristic) function $v : 2^P \to \mathbb{R}$ with $v(\emptyset) = 0$. In the case of a pooling situation, we take the player set $P = N$ and the value of v on the coalition $S \in 2^N$ will be the maximal earnings, the coalition can gain inside their possibilities i.e. $v(S)$ is the maximum of $X*E$ where $X \in \mathbb{R}_+^{N\times M}$ (or $Z_+^{N\times M}$) with the properties

$$1_S X \leq 1_S 0; \quad X 1_M \leq d*1_S. \tag{2.2}$$

Here 1_S is the vector of $\mathbb{R}^N$ with $(1_S)_i = 1$ if $i \in S$ and $(1_S)_i = 0$ otherwise and $d*1_S$ is the vector in $\mathbb{R}^N$ with coordinates $(d*1_S)_i = d_i(1_S)_i$ for all $i \in N$.

We call the set of feasible distributions for coalition S, as described by (2.2), $\Theta(S)$ and $\Theta(S)^*$ respectively. Since

$X1_M \leq d*1_S$ and $X \geq 0$ implies that $X_{ij} = 0$ for $i \notin S$, we can describe the set $\Theta(S)$ also by the constraints

$$1_N X \leq 1_S O \quad \text{and} \quad X1_M \leq d*1_S \tag{2.2a}$$

The first cooperative game related to a pooling situation <N,M,s,d,O,E> is given by $P = N$ and $v(S) = \max\{X*E \mid X \in \Theta(S)\}$ for all coalitions $S \subset N$.

There is another approach to evaluate a pooling situation. This time we forget for the time being the property rights (i.e. the matrix O) and take the set $N \cup M$ as player set. The cooperative game $\bar{v}$ gives to a coalition $S \subset N \cup M$ the maximal earning $X*E$ which can be obtained by cooperation of the members of the coalition S, neglecting all property rights. This game is typically a game with two kinds of players and can be seen as a generalization of the assignment game of Shapley/Shubik [16].
The worth $\bar{v}(s)$ of a coalition $S \subset N \cup M$ is the maximum of $X*E$ where $X \in \mathbb{R}_+^{N\times M}$ (or $Z_+^{N\times M}$) with the properties

$$1_{S\cap N} X \leq 1_{S\cap M}*s \quad \text{and} \quad X1_{S\cap M} \leq 1_{S\cap N}*d \tag{2.3}$$

We call the set of feasible distributions, as described by (2.3), $\Lambda(S)$ and $\Lambda(S)^*$ respectively. The inequality $1_{S\cap N}X \leq 1_{S\cap M}*s$ implies that $X_{ij} = 0$ if $j \notin S \cap M$ and the inequality $X1_{S\cap M} \leq 1_{S\cap N}*d$ implies that $X_{ij} = 0$ if $i \notin S \cap N$. So $\Lambda(S)$ (or $\Lambda(S)^*$) is also given by the inequalities

$$1_N X \leq 1_{S\cap M}*s \quad \text{and} \quad X1_M \leq 1_{S\cap N}*d \tag{2.3a}$$

Now we can directly prove that the cores of v and $\bar{v}$ are non-empty.

<u>Theorem 2.6</u>. The cores $C(v)$ and $C(\bar{v})$ are non-empty.

<u>Proof</u>: By the criterion of Bondareva [1] and Shapley [15] we have to prove that $\sum_{S\subset N} \lambda_S 1_S = 1_N$ with $\lambda_S \geq 0$ for all $S \subset N$ implies that $\sum_{S\in N} \lambda_S v(S) \leq v(N)$. And in the same way that

$\sum_{S \subset N \cup M} \mu_S 1_S = 1_{N \cup M}$ with $\mu_S \geq 0$ for all $S \subset N \cup M$ implies that $\sum_{S \subset N \cup M} \mu_S \bar{v}(S) \leq \bar{v}(N \cup M)$.

We shall prove that $\sum_{S \subset N} \lambda_S 1_S = 1_N$ with $\lambda_S \geq 0$ implies $\Sigma \lambda_S \Theta(S) \subset \Theta(N)$ and an analoguous statement for $\Lambda(S)$.

Let $X_S \in \Theta(S)$ for all $S \subset N$. Then $1_N(\sum_{S \subset N} \lambda_S X_S) = \sum_{S \subset N} \lambda_S 1_N X_S \leq \sum_{S \subset N} \lambda_S 1_S O = 1_N O = s$ and $(\sum_{S \subset N} \lambda_S X_S) 1_M = \sum_{S \subset N} \lambda_S (X_S 1_M) \leq \sum_{S \subset N} \lambda_S 1_S * d = = 1_N * d = d$.

The proof of $\Sigma \mu_S \Lambda(S) \subset \Lambda(N \cup M)$ proceeds in the same way.

If $v(S) = X_S * E$, then $\sum_{S \subset N} \lambda_S v(S) = (\sum_{S \subset N} \lambda_S X_S) * E \leq v(N)$, since $\sum_{S \subset N} \lambda_S X_S \in \Theta(N)$.

If the constraints s and d are integer-valued we find $\sum_{S \subset N} \lambda_S \Theta(S)^* \subset \Theta(N)$ and along the same lines of argument we find, if $v(S)^* = \max\{X * E \mid X \in \Theta(S)^*\}$,
$\Sigma \lambda_S v(S)^* \leq \max\{X * E \mid X \in \Theta(N)\}$ and by Corollary 2.5 (i)

$$\Sigma \lambda_S v(S)^* \leq \max\{X * E \mid X \in \Theta(N)^*\} = v(N)^*.$$

3. OWEN VECTORS OF POOLING SITUATIONS

In [11] Owen studied the following situation. In a linear production process there are p products which can be produced out of r resources. Given is the production matrix $A = (A_{ij})$, where A_{ij} denotes the quantity of resource $i \in R = \{1,\ldots,r\}$ which is used in the production of one unit of product $j \in P = \{1,\ldots,p\}$. Further, there is given a price vector $c \in \mathbb{R}^P_+$ of which the j-th coordinate gives the market price of one unit of product $j \in P$.

Let $Q = \{1,\ldots,q\}$ be the set of owners of the resources and for each $k \in Q$ the vector $b_k \in \mathbb{R}^R_+$ gives the quantities of the resources which player $k \in Q$ possesses.

The linear production game introduced by Owen has player set Q and the worth of a coalition S is

$$v(S) = \max\{c.x \mid x \geq 0,\ Ax \leq \sum_{k \in S} b_k\}$$

i.e. v(S) is the maximal revenue which coalition S can achieve.

The inequality $Ax \leq \sum_{k\in S} b_k$ implies that the coalition S can only use the resources which it owns.

Proposition 3.1. The cooperative games related to a pooling situation <N,M,s,d,O,E> are linear production games in the sense of Owen.

Proof: In both games the 'products' are pairs (i,j) with $i \in N$ and $j \in M$ i.e. $P = N\times M$. The 'resources' are the elements of N and the elements of N i.e. $R = N \cup M$. For the production of one unit (i,j), one unit of resource i and one unit of resource j is needed. This defines the production matrix A. The revenue arising from one unit of the product (i,j) is E_{ij}. In the first game v, related to the pooling situation <N,M,s,d,O,E>, the player set $Q = N$ and for each $k \in Q$ the resource vector is

$$b_k = (0,\ldots,d_k,0,\ldots,0;O_{1k},\ldots,O_{mk}).$$

In the second game $\bar{v}$ the player set $\bar{Q} = N \cup M$ and the resource vectors are

$$\bar{b}_i = (0,\ldots d_i\ldots 0,0,\ldots,0) \text{ if } i \in N \text{ and}$$

$$\bar{b}_j = (0,\ldots,0;0,\ldots s_j,\ldots 0) \text{ if } j \in M. \quad \square$$

In order to find core elements Owen investigates the dual problem of the linear program belonging to v(Q) i.e. minimize $\sum_{k\in Q} b_k \cdot y$ under the condition $yA \geq c$ and $y \geq 0$. A solution $\hat{y}$ of this linear program can be seen as a shadow price vector for the resources and Owen proves that the vector $x = (x_1 \ldots x_q)$ with $x_k = b_k \cdot \hat{y}$ for all $k \in Q$ is a core element. Note that x_k is the value of the resource vector b_k of player k under the price vector $\hat{y}$.
The core elements, in this way obtained, we shall call the Owen vectors of the production situation.

Example 3.2. Consider the production situation with production matrix $\binom{1}{1}$, two players 1 and 2, price vector $c > 0$, and resource vectors $b_1 = (1,0)$, $b_2 = (0,2)$.
Then the linear production game has the following values

$v(1) = v(2) = 0$ and $v(1,2) = c$.

The dual programm to be considered is: minimize y_1+2y_2 under the conditions $y_1 \in \mathbb{R}_+$, $y_2 \in \mathbb{R}_+$ and $y_1+y_2 \geq c$. There is only one solution $y_1 = c$ and $y_2 = 0$ and this gives the Owen vector $(c,0)$.

If we interchange the vectors b_1 and b_2 we find the same cooperative game but now the Owen vector is $(0,c)$. The Owen vector gives account for the relative scarcity of resource 1 in the first production situation and of resource 2 in the second production situation.

The cooperative games related to the production situations make no distinction between both production situations.

We compute now the Owen vectors of the two cooperative games v and $\bar{v}$ related to a pooling situation. Notice that $\Theta(N) = \Lambda(N \cup M)$ and $v(N) = \bar{v}(N \cup M)$.

The dual program of the linear program which determines $v(N) = \bar{v}(N \cup M)$ is as follows. Minimize $y.d + z.s$ under the conditions $y \in \mathbb{R}_+^N$, $z \in \mathbb{R}_+^M$ and $y_i+z_j \geq E_{ij}$ for all pairs $(i,j) \in N\times M$. If $(\hat{y},\hat{z})$ is a solution of the dual program, this shadow price vector attributes in the first game to player $i \in N$ the amount $\hat{x}_i = d_i\hat{y}_i + \sum_{j\in M} O_{ij}\, \hat{z}_j$.

The same shadow price gives in the second game to player $i \in N$ the amount $\hat{x}_i' = d.\hat{y}_i$ and to player $j \in M$ the amount $\hat{x}_j'' = s_j.\hat{z}_j$. So we find the following consistency relation between the Owen vectors $\hat{x}$ of the first game and the Owen vector $(\hat{x}',\hat{x}'')$ of the second game under the same price vector

$$\hat{x} = \hat{x}'+O'\hat{x}''$$

where O_{ij}' is the fraction of s_j which player i possessess i.e. $O_{ij}' = O_{ij}s_j^{-1}$ if $s_j > 0$. So, if Ow is the set of Owen vectors of v and $\overline{Ow}$ is the set of Owen vectors of $\bar{v}$, then

<u>Theorem 3.3</u>. $Ow = (I,O')\overline{Ow}$ where I is the identity matrix of size n. □

<u>Remarks</u>: The way in which in the second cooperative game $\bar{v}$ the component $\hat{x}''$ attibuted to the commodities, is divided among the owners of the commodities is a very natural one.

If, for example, in example 2.1, hotel (owner) H_j obtains $\hat{x}_j''$ by the Owen distribution and two travel agencies made reservations for all the rooms of hotel H_j in the proportion 40:60, it seems natural to divide $\hat{x}_j''$ in the same proportion and to give these amounts to the two travel agencies. This method is so natural that one may wonder whether other solution concepts may have the same property. The next example shows that this is not the case for the core, the Shapley value, the nucleolus and the τ-value. For the definition and the most elementary properties of these solution concepts we refer to [14] for the Shapley value, to [13] for the nucleolus and to [17] for the τ-value.

Example 3.4. Let $N = \{1,2\}$, $M = \{I,II\}$ and $E = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$. Further $s = d = (1,1)$ and $O(= O') = \begin{bmatrix} q & \bar{q} \\ 1-q & 1-\bar{q} \end{bmatrix}$ with $q,\bar{q} \in [0,1]$. The cooperative game v, related to this pooling situation has values

$$v(1) = 2 \min (1,q+\bar{q}),\ v(2) = \min (1,2-q-\bar{q}) \text{ and } v(1,2) = 3.$$

The Shapley value, the nucleolus and the τ-value of this 2-person game coincide and has the first coordinate

$$t = \tfrac{1}{2}v(1,2)+\tfrac{1}{2}v(1)+\tfrac{1}{2}v(2) = \frac{3}{2} + \min(1,q+\bar{q}) - \tfrac{1}{2}\min(1,2-q-\bar{q}) =$$

$$= \begin{cases} \frac{3}{2} + \frac{1}{2}(q+\bar{q}) & \text{if } q+\bar{q} \geq 1 \\ 1 + (q+\bar{q}) & \text{if } q+\bar{q} \leq 1 \end{cases} = \min(\tfrac{3}{2} + \tfrac{1}{2}(q+\bar{q});1+q+\bar{q}).$$

The second coordinate is $s = 3-t$.

Further, $(t,3-t)$ is an element of the core $C(v)$ iff

$t \geq 2\min(1,q+\bar{q})$ and

$$t \leq 3 - \min(1,2-q-\bar{q}) \text{ i.e. } \begin{cases} 2 \leq t \leq 1+q+\bar{q} & \text{if } q+\bar{q} \geq 1 \\ 2(q+\bar{q}) \leq t \leq 2 & \text{if } q+q \leq 1 \end{cases}.$$

The second cooperative game $\bar{v}$ is the smallest monotonic game with

$\bar{v}(1,I) = \bar{v}(1,II) = 2$, $\bar{v}(2,I) = \bar{v}(2,II) = 1$ and

$\bar{v}(1,2,I,II) = 3$.

The Shapley value $\phi(\bar{v})$ can be computed to be $(\frac{7}{6},\frac{1}{2},\frac{2}{3},\frac{2}{3})$.

The core of this game is $C(\bar{v}) = \{(t+1,t,1-t,1-t) \mid t \in [0,1]\}$.

Since the nucleolus $n(\bar{v})$ is in the core, we know already $n(v) = (t+1,t,1-t.1-t)$ for some $t \in [0,1]$. The excesses of the coalitions $\{1,I\}$, $\{1,II\}$, $\{2,I\}$ and $\{2,II\}$ are zero. Further we find the excesses: $e(1) = -t-1$, $e(2) = -t$, $e(I) = e(II) = t-1$.

$$e(1,2,I) = e(1,2,II) = -t \text{ and } e(1,I,II) = e(2,I,II) = t-1.$$

The largest negative excess is $\max(-t;t-1)$ and this excess is minimal if $t = \frac{1}{2}$. Hence $n(\bar{v}) = (\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$. Computing $\tau(\bar{v})$, we find $M(\bar{v}) = (2,1,1,1)$ and $m(\bar{v}) = (1,0,0,0)$ as upper and lower bound. So $\tau(\bar{v}) = (\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}) = n(\bar{v})$.
Take $q+\bar{q} = 0.9$. Then we find:

$$(I,O)C(\bar{v}) = \{(1.9+0.1t,1.1-0.1t) \mid t \in [0,1]\} \subsetneqq$$
$$C(v) = \{(1.8+0.2t,1.2-0.2t) \mid t \in [0,1]\}$$
$$(I,O)\phi(\bar{v}) = (\tfrac{53}{30},\tfrac{37}{30}) \neq (1.9;1,1) = \phi(v)$$
$$(I,O)n(\bar{v}) = (I,O)\tau(\bar{v}) = (1.95;1.05) \neq (1.9;1.1) = n(v) = \tau(v)$$

4. RELATED TOPICS IN THE LITERATURE

(1) Permutation games: In the introduction we already mentioned that situations which give rise to permutation games are examples of pooling situations. To make this remark more precise, we obtain the pooling situation with N = M, E and $O = I_N$, the identity matrix. Further $s = d = (1,\dots,1)$. The first game related to this pooling situation is just the permutation game of [3] and [18] (where costs are replaced by earnings).
According to a letter of Shapley, which we mentioned in [18], permutation games can be considered as assignment games (Shapley/Shubik [16]) by splitting up each player into a selling and a buying player. The assignment game we get in this case, is the second cooperative game $\bar{v}$ related to the pooling situation. To find core elements of the permutation games Shapley essentially proposed to take a core element (= Owen vector) of the assignment game and to apply $(I,O) = (I,I)$ to this vector, just as we did in theorem 3.3.

(2) Assignment games: Assignment situations as in [16] fail to be pooling situations by the absence of ownership relations. The second approach of pooling situations, the cooperative game $\bar{v}$, neglects the ownership relations and can therefore be considered as a generalization of the assignment games of Shapley/Shubik [16] in the sense that sellers and buyers may dispose or may want to dispose of more than one item of the indivisible goods.
In a perhaps artificial way we can reform a generalized assignment situation into a pooling situations by taking N and M equal to the union of the set of sellers and the set of buyers. Further we put

$$E = \begin{bmatrix} \emptyset & A \\ \emptyset & \emptyset \end{bmatrix} \quad \text{and} \quad O = \begin{bmatrix} d_1 & & & & & \\ & \ddots & & & \emptyset & \\ & & d_n & & & \\ & & & s_1 & & \\ & \emptyset & & & \ddots & \\ & & & & & s_m \end{bmatrix}$$

where $A \geq 0$ is the matrix belonging to the assignment game. In [3] this has been studied for non-generalized assignment games.

(3) Job matching: This problem studied in Crawford/Knoer [2] is a kind of assignment problem and all that has been said about these problems, can be done here.

(4) In the N.T.U. sphere the papers of Kaneko [7] , [8] and of Quinzii [12] should be mentioned. In particular, the last paper can easily be extended as to comprehend pooling situations. The initial endowment with indivisible goods can replace the property right matrix O. The utility functions should be defined on pairs of money and a commodity bundle consisting of a non-negative number of items of different commodities.
In particular, the question under what conditions <u>all</u> core elements are price equilibria may be very interesting.

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